Mathematical Modeling

Information or data can be modeled mathematically. A mathematical model is much like any other model, it is a simplified representation of reality. Like all models, mathematical models are never perfectly accurate. Since they simplify reality they also change it. When children play with dolls or model cars they are learning valuable skills. These models are useful to them, however, trying to learn to perform surgery using a doll or learning auto mechanics from a model car would reveal that the models are not perfect. That doesn't mean they aren't useful, as long as we know their limitations.

Mathematical models come in several forms: data tables, graphs, equations, words, system dynamics models and sometimes others like geometric models. We are going to learn to create and use these kinds of models in order to understand our world better.

Let's imagine someone driving a car, they check their trip odometer every hour of driving and get the following data:

Time (hours)	Distance (miles)
1	63
2	121
3	178
4	242
5	301

How could we model this data, that is, how could we simplify it so it is easier to understand and work with?

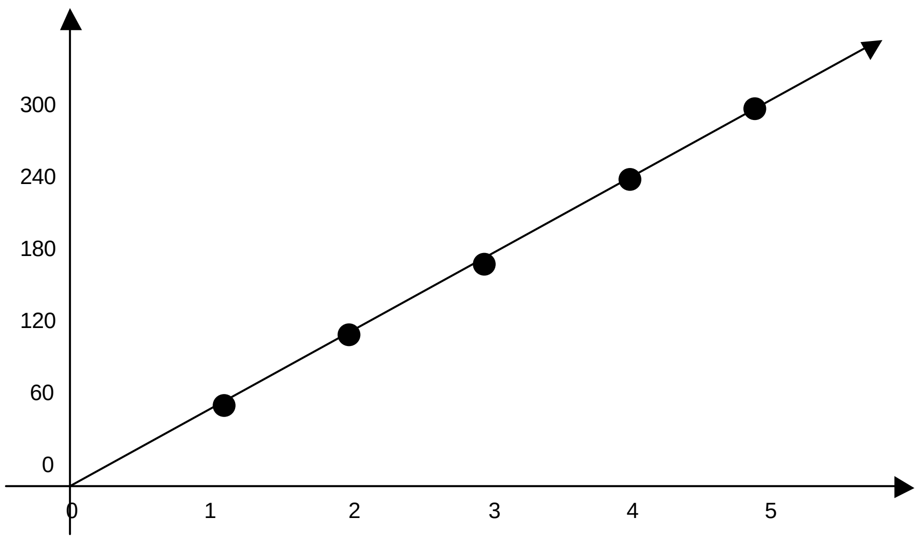
WORDS: Well, we could say this person had been driving about 60 miles per hour.

EQUATION: We could write an equation that is a shorter way of saying what we said in words: $d = 60t$. We would say the initial conditions were 0 miles and that the rate of change was 60 mph.

DATA TABLE: We could create a data table that approximates the data in an easier to understand way. Notice that the data in this table does not match the original data but approximates it pretty well.

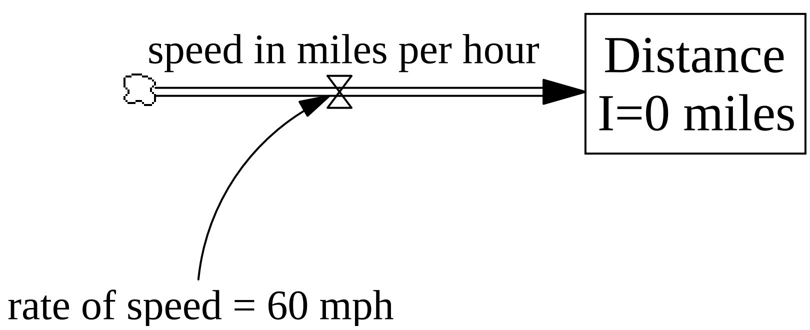
Time (hours)	Distance (miles)
0	0
1	60
2	120
3	180
4	240
5	300

GRAPH: We can graph the original data by placing the data points on a graph. Then we can approximate the data by drawing a line that smoothes the way between them. This line is call a trend line or a line of best fit or a regression line.



This line has an intercept of 0 miles and a slope of 60 mph. Notice that these words refer to the same numbers as the initial condition and rate of change did in the equation.

SYSTEM DYNAMICS MODEL:



This model simulates the distance traveled by imagining it to be like water flowing. The distance is accumulating like water in a tank. The initial amount in the tank, I, is 0. The distance is changing at a certain speed much like water flows into a tank at a certain speed. A valve on the inflow pipe controls the speed at which the water flows into the tank. The rate of speed controls the valve. The rate of speed is set at 60 mph.

All of these models accomplish the same thing. They give us simpler way to think about the real situation we analyzing.

Interpolating: Creating models allows us to interpolate the data. This means we can estimate where the car was between the points we already know. If we want to know how far the car had gone after two and a half hours we can use the equation, graph or the table to tell us. The equation tells us that to find the distance we multiply 60 by the time traveled. Since 60 * 2.5 is 150, the distance must be about 150 miles. We just look about half way between 2 and 3 hours and estimate the distance. It looks like it should be about half way between 120 and 180 miles or about 150 miles. In order to interpolate using the system dynamics model you would need to draw the model in a computer program like Vensim or STELLA and run the model to create a graph or table. Vensim PLE is available free from Vensim at http://www.vensim.com . More information about system dynamics and modeling is available at http://sysdyn.clexchange.org/road-maps .

Extrapolating: Creating models also allows us to extrapolate the data. This means we can estimate where the car was beyond the points we already know. If we want to know how far the car would have gone after six hours we can use the equation, graph or the table to tell us the same way. The equation tells us that to find the distance we multiply 60 by the time traveled. Since 60 * 6 is 360, the distance must be about 360 miles. With the table or graph we just to 6 hours in our minds and estimate the distance. It looks like it should be 60 miles further than 300 miles or about 360 miles. In order to extrapolate using the system dynamics model you would need to draw the model in a computer program like Vensim or STELLA and run the model to create a graph or table.

INVERSE FUNCTIONS: Models are sometimes called functions. Models or functions can also be used to interpolate or extrapolate in reverse. In our example, it might be useful to know at what time some number of miles would have been traveled. Using the table or the graph we could reverse the procedure we used earlier. If we wanted to know how long it would take to travel 210 miles we could look up 210 miles and interpolate and estimate that it would take three and a half hours.

Using the equation $d = 60t$, we can find the inverse function by solving the equation for t.

$$d = 60t$$

$$\frac{d}{60} = \frac{60t}{60}$$

$$\frac{d}{60} = t$$

$$t = \frac{d}{60}$$

In this case since the t is being multiplied by 60, we do the opposite and divide by 60. Now t is both multiplied and divided by 60 so it remains unchanged or just t. Since $d = 60t$, d is the same as 60t, since we divided 60t by 60 we must also divided what it is equal to, the d, by 60. This way the two things remain equal.

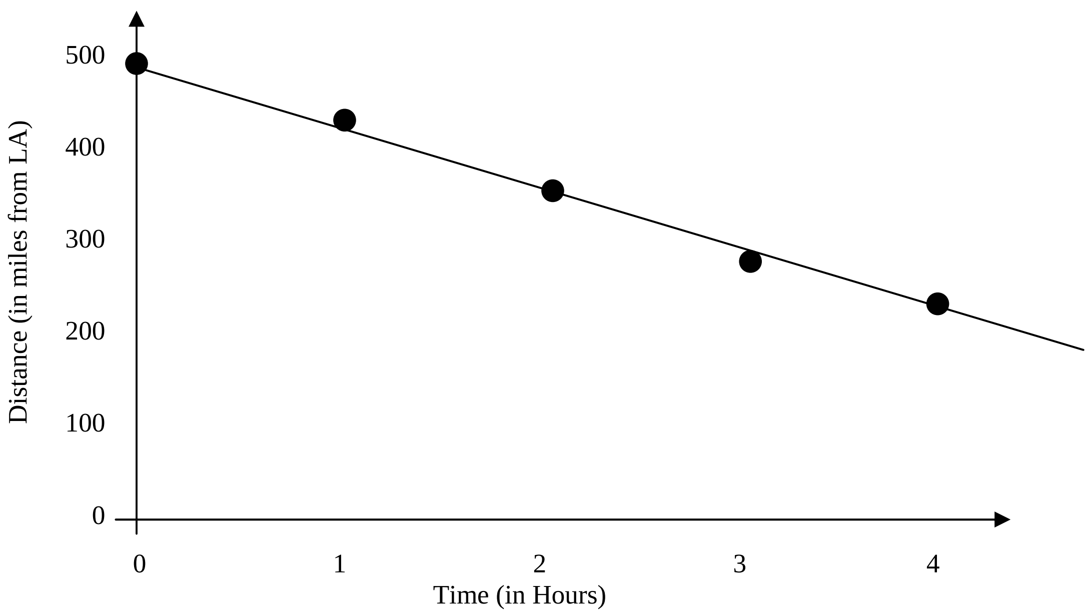
Now our new equation, the inverse function, says: $t = \frac{d}{60}$, that is to find out at what time a certain distance has been traveled take that distance and divide it by 60. In our example 210 miles divided by 60 mph is 3.5 hours. Usually equations are faster and easier to use that tables or graphs. We find all inverse functions by this same equation solving process.

Linear Modeling

What if the situation were a bit more complicated? What if our starting point were not 0? What if we were driving to Los Angeles, initially 500 miles away? Our data table for the first day of this trip might look like this:

Time (hours)	Distance (miles)
0	500
1	436
2	363
3	290
4	220

If we were to graph this data we would get something like this:

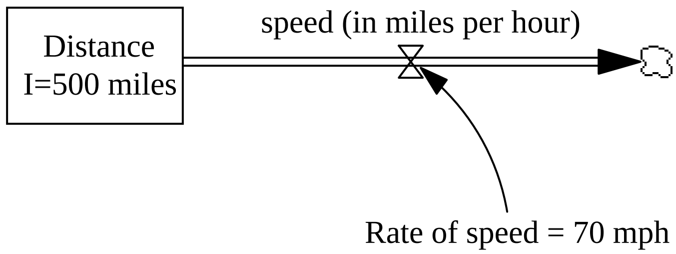


Notice that the dots representing our data don't exactly match the line. This is showing us graphically that the model doesn't match the data perfectly, remember, that's OK.

How would we describe this situation in words? Well, we started out 500 miles from L.A. and got closer to it by about 70 miles every hour. Where did the 70 come from? One way to get it would be to notice that we went about 280 miles in 4 hours which is an average of 70 miles per hour.

How would we abbreviate these words with an equation? We could write d = 500 – 70t. That is, our distance, d, started at 500 miles and decrease 70 miles for every hour of driving, t.

How would we draw the system dynamics model for this situation. Well, much like the last one. We are imagining water draining out of a tank at a constant rate.



The tank starts out with 500 gallons and drains at 70 gallons per hour. We change the labels from gallons to miles and we have a circumstance that matches the one we want to model.

How would we find the inverse function?

Start by writing the equation. $d = 500 - 70t$

Subtract the initial condition from both sides. $d - 500 = -70t$

Divide both sides by –70. $$\frac{d - 500}{-70} = \frac{-70t}{-70}$$

Since the t is multiplied and divided by –70 it is just t, so our inverse function is:

$$t = \frac{d - 500}{-70}$$

We can use this to calculate that our t will be 0, that is, we will arrive in L.A., after –500/-70 hours. Since –500/-70 is a bit over seven hours we know that this is how long it will take us to get to L.A.